

Prepared in partial fulfilment of the Study Oriented Project

Course No: MATH F266  
  
AT  
  
**KK BIRLA, BITS PILANI, GOA CAMPUS**

Consolidated Project Report

**Reimann Problem for Hyperbolic**

**System of Partial Differential**

**Equations**

VIDIT GOYAL-2021B4AA2944G

KALASH PODDAR – 2021B4AA3025G

LAKSHAY SHEWANI – 2021B4AA2786G

**ABSTRACT**

The "Riemann Problem for Hyperbolic Systems of Partial Differential Equations" poses a significant challenge in computational mathematics and fluid dynamics. This project focuses on investigating and comparing four prominent numerical schemes—Lax-Friedrichs, Lax-Wendroff, Warming-Beam, and Godunov—for solving this complex problem. Each scheme offers distinct advantages and limitations in approximating solutions to hyperbolic systems, motivating a comparative analysis to discern their efficacy across various scenarios. Employing rigorous mathematical formulations and computational simulations, this study delves into the behavior of these schemes under diverse conditions, aiming to elucidate their respective strengths and weaknesses. Through meticulous examination of numerical accuracy, stability, and computational efficiency, this research contributes to a deeper understanding of numerical methods for hyperbolic systems. By synthesizing theoretical insights with computational results, this investigation provides valuable insights into the practical applicability of these schemes, paving the way for informed decision-making in computational fluid dynamics and related fields.

# Table of Contents

Introduction 2

Objectives and Scope 2

Background 2

Notions on Hyperbolic Partial Differential Equations 5

1.1 Quasi-Linear Equations: Basic Concepts 5

1.2 The Linear Advection Equation 5

1.2.1 The Riemann Problem 5

1.3 Linear Hyperbolic Systems 5

1.3.1 Diagonalisation and Characteristic Variables 5

1.3.2 Some Useful Definitions 5

1.3.3 Shock Waves 5

One Dimensional Euler Equations 15

2.1 Conservative Formulation 15

2.2 Non Conservative Formulation 15

2.2.1 Primitive Variable Formulation 15

2.3 Elementary Wave Solutions to Reimann Problem 15

2.3.1 Contact Discontinuities 15

2.3.2 Rarefaction Waves 15

The Riemann Problem for the Euler Equations 27

3.1 Equations for Pressure and Particle Velocity 27

3.1.1 Function fL for a Left Shock 27

3.1.2 Function fL for Left Rarefaction 27

3.2 Numerical Tests 27

Numerical Solution Using Iterative Schemes 35

4.1 Some Well-Known Schemes 35

4.2 Godunov's Method 35

4.3 Sample Numerical Results 35

4.3.1 Linear Advection 35

4.3.2 The Inviscid Burgers Equation 35

Conclusion 48

Summary of Findings 48

Future Work 48

References 52

Appendices 54

Additional Calculations 54

Source Code 54

**1: Notions on Hyperbolic Partial Differential Equations**

**1.1 Quasi-Linear Equations: Basic Concepts**

In this section we study systems of first-order partial differential equations of the form

for . This is a system of equations in unknowns that depend on space and a time-like variable . Here are the dependent variables and are the independent variables; this is expressed via the notation denotes the partial derivative of with respect to ; similarly denotes the partial derivative of with respect to . We also make use of subscripts to denote partial derivatives. System (1.1) can also be written in matrix form as

with

If the entries of the matrix are all constant and the components of the vector are also constant then system (1.2) is linear with constant coefficients. If and the system is linear with variable coefficients. The system is still linear if depends linearly on and is called quasi-linear if the coefficient matrix is a function of the vector , that is . Note that quasi-linear systems are in general systems of non-linear equations. System (1.2) is called homogeneous if . For a set of PDEs of the form (1.2) one needs to specify the range of variation of the independent variables and . Usually lies in a subinterval of the real line, namely ; this subinterval is called the spatial domain of the PDEs, or just domain. At the values one also needs to specify Boundary Conditions (BCs). In this Chapter we assume the domain is the full real line, , and thus no boundary conditions need to be specified. As to variations of time we assume . An Initial Condition (IC) needs to be specified at the initial time, which is usually chosen to be .

Two scalar examples of PDEs of the form (1.1) are the linear advection equation

and the inviscid Burgers equation

In the linear advection equation (1.4) the coefficient (a constant) is the wave propagation speed. In the Burgers equation .

**Definition 1.1 (Conservation Laws):** Conservation laws are systems of partial differential equations that can be written in the form

where

is called the vector of conserved variables, is the vector of fluxes and each of its components is a function of the components of .

**Definition 1.2 (Jacobian Matrix):** The Jacobian of the flux function in (1.6) is the matrix

The entries of are partial derivatives of the components of the vector with respect to the components of the vector of conserved variables , that is .

Note that conservation laws of the form (1.6)-(1.7) can also be written in quasi-linear form (1.2), with , by applying the chain rule to the second term in (1.6), namely

Hence (1.6) becomes

which is a special case of (1.2). The scalar PDEs (1.4) and (1.5) can be expressed as conservation laws, namely

**Definition 1.3 (Eigenvalues):** The eigenvalues of a matrix are the solutions of the characteristic polynomial

where is the identity matrix. The eigenvalues of the coefficient matrix of a system of the form (1.2) are called the eigenvalues of the system.

Physically, eigenvalues represent speeds of propagation of information. Speeds will be measured positive in the direction of increasing and negative otherwise.

**Definition 1.4 (Eigenvectors):** A right eigenvector of a matrix A corresponding to an eigenvalue of is a vector satisfying . Similarly, a left eigenvector of a matrix corresponding to an eigenvalue of is a vector such that .

For the scalar examples the eigenvalues are trivially found to be and respectively. Next we find eigenvalues and eigenvectors for a system of PDEs.

*Example 2.1 (Linearized Gas Dynamics):* The linearized equations of Gas Dynamics are the linear system

where the unknowns are the density and the speed ; is a constant reference density and is the sound speed, a positive constant. When written in the matrix form (1.2) this system reads

with

The eigenvalues of the system are the zeros of the characteristic polynomial

That is, , which has two real and distinct solutions, namely

We now find the right eigenvectors corresponding to the eigenvalues and .

The eigenvector for eigenvalue is found as follows: we look for a vector such that is a right eigenvector of , that is . Writing this in full gives

which produces two linear algebraic equations for the unknowns and

The reader will realise that in fact these two equations are equivalent and so effectively we have a single linear algebraic equation in two unknowns. This gives a one-parameter family of solutions. Thus we select an arbitrary nonzero parameter , a scaling factor, and set in any of the equations to obtain for the second component and hence the first right eigenvector becomes

The eigenvector for eigenvalue is found in a similar manner. The resulting algebraic equations for corresponding to the eigenvalue are

By denoting the second scaling factor by and setting we obtain

Taking the scaling factors to be and gives the right eigenvectors

**Definition 1.6 (Hyperbolic System)**: A system (1.2) is said to be hyperbolic at a point if has real eigenvalues and a corresponding set of linearly independent right eigenvectors . The system is said to be strictly hyperbolic if the eigenvalues are all distinct.

Note that strict hyperbolicity implies hyperbolicity, because real and distinct eigenvalues ensure the existence of a set of linearly independent eigenvectors. The system is said to be elliptic at a point if none of the eigenvalues of are real. Both scalar examples (1.9)-(1.10) are trivially hyperbolic. The linearised gas dynamic equations (1.12) are also hyperbolic, since and are both real at any point . Moreover, as the eigenvalues are also distinct this system is strictly hyperbolic.

**1.2 The Linear Advection Equation**

A general, time-dependent linear advection equation in three space dimensions reads

where the unknown is and are variable coefficients. If the coefficients are sufficiently smooth one can express (1.21) as a conservation law with source terms, namely

In this section we study in detail the initial-value problem (IVP) for the special case of the linear advection equation, namely

where is a constant wave propagation speed. The initial data at time is a function of alone and is denoted by . We warn the reader that for systems we shall use a different notation for the initial data. Generally, we shall not be explicit about the conditions on the independent variables when stating an initial-value problem. The PDE in (1.23) is the simplest hyperbolic PDE and in view of (1.9) is also the simplest hyperbolic conservation law. It is a very useful model equation for the purpose of studying numerical methods for hyperbolic conservation laws, in the same way as the linear, first-order ordinary differential equation

is a popular model equation for analyzing numerical methods for Ordinary Differential Equation (ODEs).

**1.2.1 The Riemann Problem**

By using geometric arguments we have constructed the analytical solution of the general IVP (1.23) for the linear advection equation. Now we study a special IVP called the Riemann problem

where (left) and (right) are two constant values, as shown in Fig. 1.1. Note that the initial data has a discontinuity at . IVP is the simplest initial-value problem one can pose. The trivial case would result when . From the previous discussion on the solution of the general IVP (1.23) we expect any point on the initial profile to propagate a distance in time . In particular, we expect the initial discontinuity at to propagate a distance in time . This particular characteristic curve will then separate those characteristic curves to the left, on which the solution takes on the value , from those curves to the right, on which the solution takes on the value ; see Fig. 1.2. So the solution of the Riemann problem (1.25) is simply

Solution (1.26) also follows directly from the general solution namely

. From (1.25), if

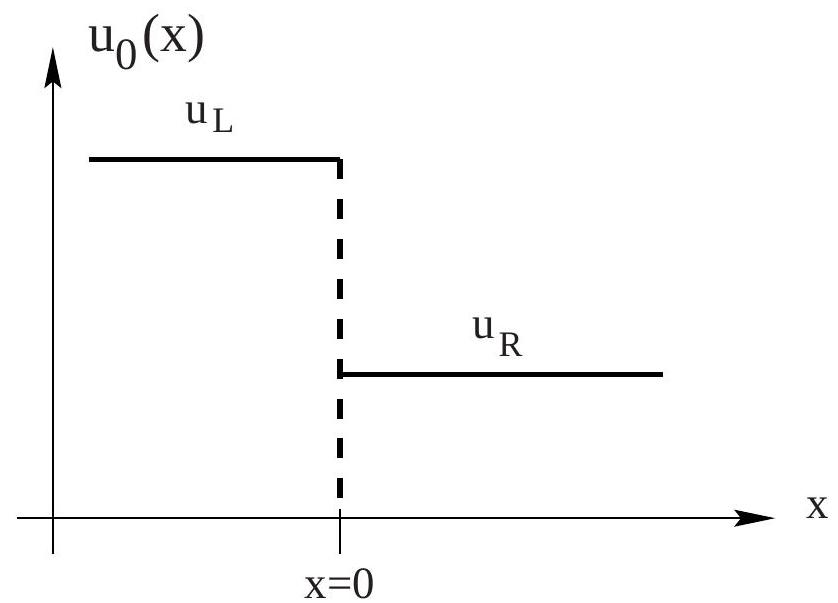


Fig. 1.1. Illustration of the initial data for the Riemann problem. At the initial time the data consists of two constant states separated by a discontinuity at

and if . The solution of the Riemann problem can be represented in the plane, as shown in Fig. 1.1. Through any point on the -axis one can draw a characteristic. As a is constant these are all parallel to each other. For the solution of the Riemann problem the characteristic that passes through is significant. This is the only one across which the solution changes.

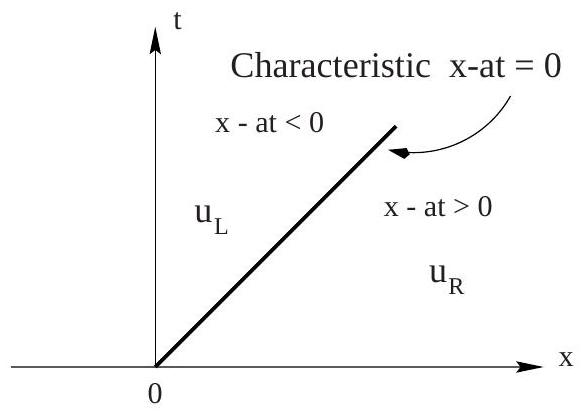


Fig. 1.2. Illustration of the solution of the Riemann problem in the plane for the linear advection equation with positive characteristic speed

**1.3 Linear Hyperbolic Systems**

In the previous section we studied in detail the behaviour and the general solution of the simplest PDE of hyperbolic type, namely the linear advection  
equation with constant wave propagation speed. Here we extend the analysis to sets of hyperbolic PDEs of the form

where the coefficient matrix is constant. From the assumption of hyperbolicity has real eigenvalues and linearly independent eigenvectors .

**1.3.1 Diagonalisation and Characteristic Variables**

In order to analyse and solve the general IVP for (1.27) it is found useful to transform the dependent variables to a new set of dependent variables . To this end we recall the following definition

**Definition 1.7 (Diagonalisable System):** A matrix A is said to be diagonalisable if can be expressed as

in terms of a diagonal matrix and a matrix . The diagonal elements of are the eigenvalues of and the columns of are the right eigenvectors of corresponding to the eigenvalues , that is

A system (1.27) is said to be diagonalisable if the coefficient matrix is diagonalisable. Based on the concept of diagonalisation one often defines a hyperbolic system (1.27) as a system with real eigenvalues and diagonalisable coefficient matrix.

**1.3.2 Some Useful Definitions**

Next we recall some standard definitions associated with hyperbolic systems

**Definition 1.8 (Domain of Dependence):** Recall that for the linear advection equation the solution at a given point depends solely on the initial data at a single point on the -axis. This point is obtained by tracing back the characteristic passing through the point . As a matter of fact, the solution at is identical to the value of the initial data at the point . One says that the domain of dependence of the point is the point . For a system the domain of dependence is an interval on the -axis that is subtended by the characteristics passing through the point .

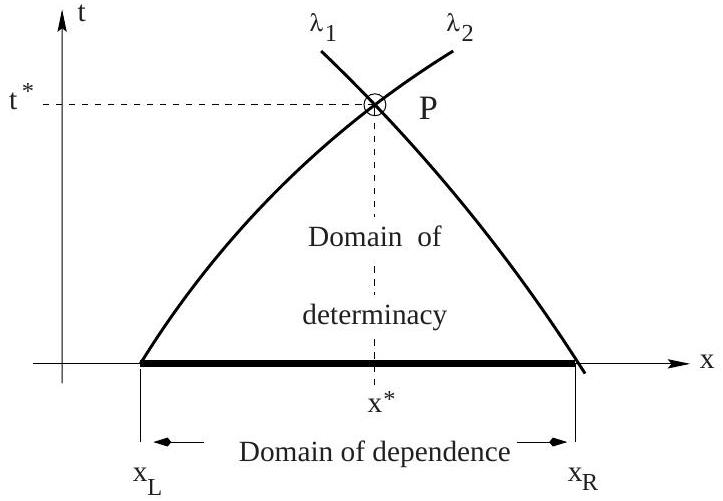


Fig. 1.3. Domain of dependence of point and corresponding domain of determinacy, for a 2 by 2 system

Fig. 1.3 illustrates the domain of dependence for a system with characteristic speeds and , with . In general, the characteristics of a hyperbolic system are curved. For a larger system the domain of dependence is determined by the slowest and fastest characteristics and is always a bounded interval, as the characteristic speeds for hyperbolic systems are always finite.

**Definition 1.9 (Domain of Determinacy).** For a given domain of dependence , the domain of determinacy is the set of points , within the domain of existence of the solution , in which is solely determined by initial data on .

In Fig. 1.3 we illustrate the domain of determinacy of an interval for the case of a system with characteristic speeds and , with .

Definition 1.17 (Range of Influence). Another useful concept is that of the range of influence of a point on the -axis. It is defined as the set of points in the plane in which the solution is influenced by initial data at the point .

Fig. 1.4 illustrates the range of influence of a point for the case of a system with characteristic speeds and , with .

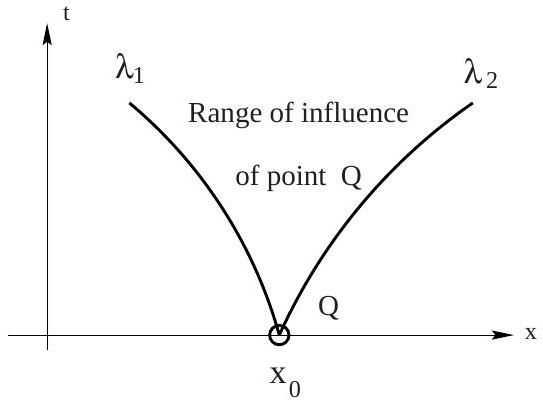


Fig. 1.4. Range of influence of point for a 2 by 2 system

has real eigenvalues and a complete set of linearly independent eigenvectors , which we assume to be ordered as

**Shock Waves**

Shock waves in air are small transition layers of very rapid changes of physical quantities such as pressure, density and temperature. The transition layer for a strong shock is of the same order of magnitude as the mean-free path of the molecules, that is about . Therefore replacing these waves as mathematical discontinuities is a reasonable approximation. Very weak shock waves such as sonic booms, are an exception; the discontinuous approximation here can be very inaccurate indeed.

We therefore insist on using the simplified model but in its integral formConsider a solution such that and their derivatives are continuous everywhere except on a line on the plane across which has a jump discontinuity. Select two fixed points and on the -axis such that . Enforcing the conservation law in integral form on the control volume leads to

where is the limit of as tends to from the left, is the limit of as tends to from the right and is the speed of the discontinuity. As is bounded the integrals vanish identically as is approached from left and right and we obtain

This algebraic expression relating the jumps , and the speed of the discontinuity is called the Rankine-Hugoniot condition and is usually expressed as

For the scalar case considered here one can solve for the speed as

Therefore, in order to admit discontinuous solutions we may formulate the problem in terms of PDEs, which are valid in smooth parts of the solution, and the Rankine-Hugoniot conditions across discontinuities.

Fig. 1.4. However, this solution is physically incorrect. The discontinuity has not arisen as the result of compression, ; the characteristics diverge from the discontinuity. This solution is called a rarefaction shock, or entropy-violating shock, and does not satisfy the entropy condition it is therefore rejected as a physical solution.

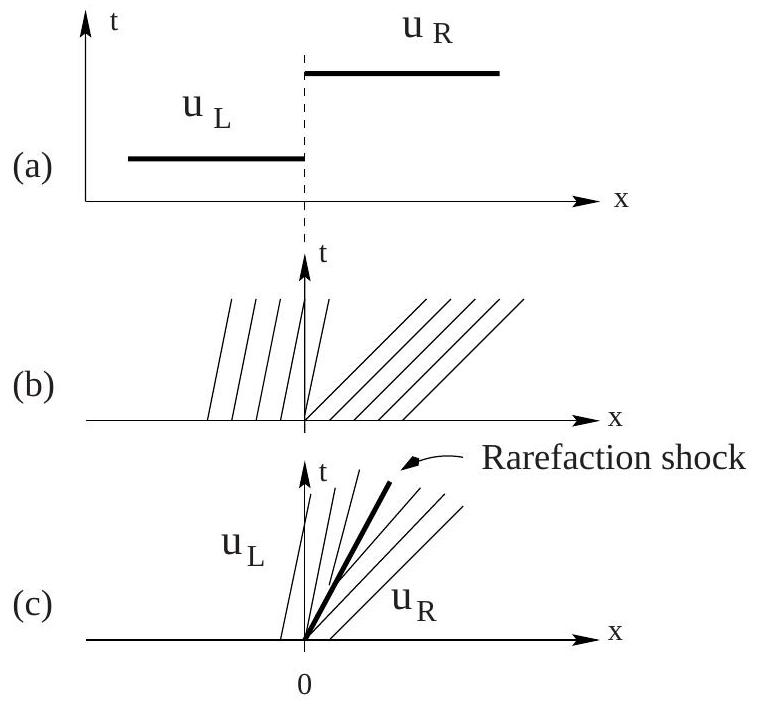


Fig. 1.4. (a) Expansive discontinuous initial data (b) picture of characteristics and (c) rarefaction shock solution on - plane

**Rarefaction Waves**

Reconsider the IVP (2.100) with general convex flux function

and expansive initial data, . As discussed previously, the entropyviolating solution to this problem is

Amongst the various other reasons for rejecting this solution as a physical solution, instability stands out as a prominent argument. By instability it is meant that small perturbations of the initial data lead to large changes in the solution. As a matter of fact, under small perturbations, the whole character of the solution changes completely, as we shall see.

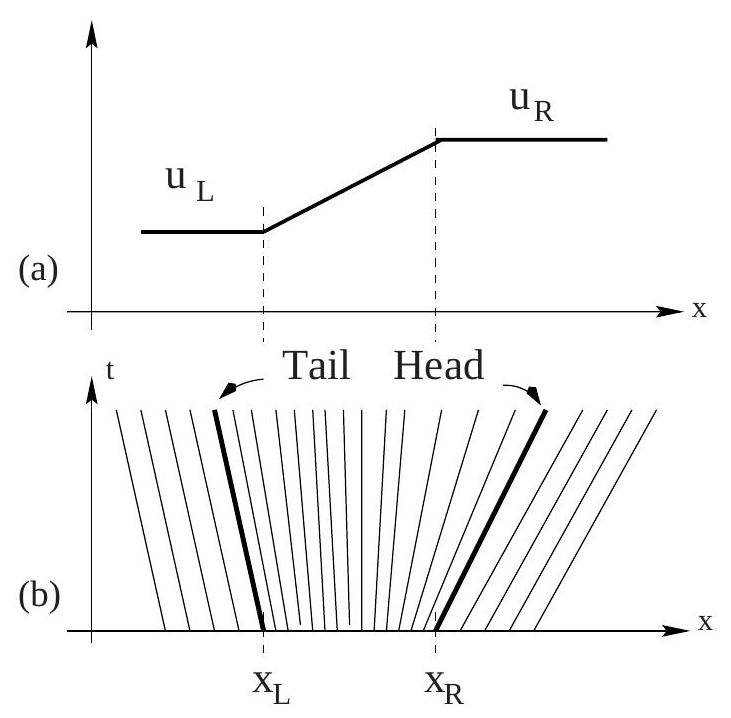
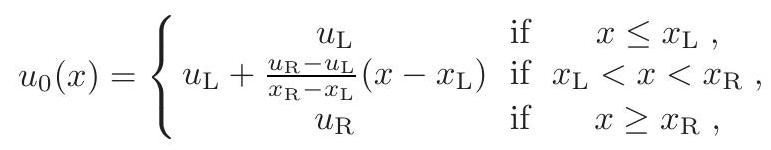


Fig. 1.5. Non-centred rarefaction wave: (a) expansive smooth initial data, (b) picture of characteristics on plane

Let us modify the initial data in (1.33) by replacing the discontinuous change from to by a linear variation of between two fixed points and . Now the initial data reads



and is illustrated in Fig. 1.5a. The corresponding picture of characteristics emanating from the initial time is shown in Fig. 1.5b. The solution to this problem is found by following characteristics, as discussed previously, and consists of two constant states, and , separated by a region of smooth transition between the data values and . This is called a rarefaction wave. The right edge of the wave is given by the characteristic emanating

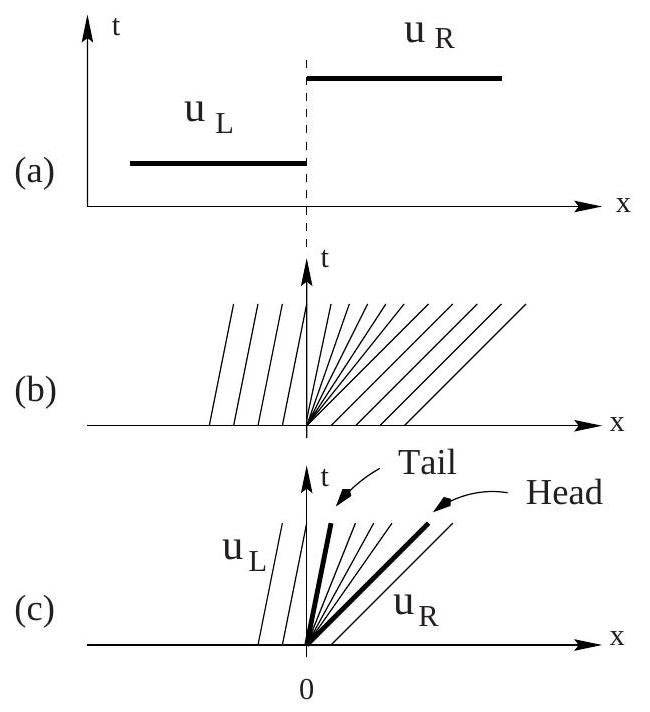


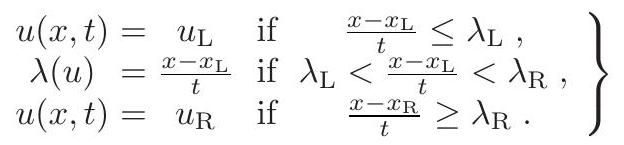
Fig. 1.6. Centred rarefaction wave: (a) expansive discontinuous initial data (b) picture of characteristics (c) entropy satisfying (rarefaction) solution on plane

from

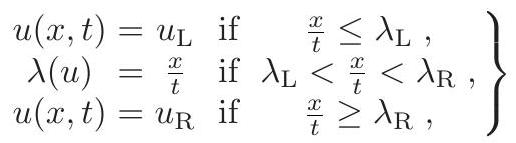
and is called the Head of the rarefaction. It carries the value . The left edge of the wave is given by the characteristic emanating from

and is called the Tail of the rarefaction. It carries the value .

As we assume convexity, , larger values of propagate faster than lower values and thus the wave spreads and flattens as time evolves. The spreading of waves is a typical non-linear phenomenon not seen in the study of linear hyperbolic systems with constant coefficients. The entire solution is



No matter how small the size of the interval over which the discontinuous data in IVP (1.33) has been spread over, the structure of the above rarefaction solution remains unaltered and is entirely different from the rarefaction shock solution (1.34), for which small changes to the data lead to large changes in the solution. Thus the rarefaction shock solution is unstable. From the above construction the rarefaction solution is stable and as and approach zero from below and above respectively, the discontinuous data at in IVP (1.33) is reproduced. Therefore, the limiting case is to be  
interpreted as follows: takes on all the values between and at and consequently takes on all the values between and at . As higher values propagate faster than lower values the initial data disintegrates immediately giving rise to a rarefaction solution. This limiting rarefaction in which all characteristics of the wave emanate from a single point is called a centred rarefaction wave. The solution is



and is illustrated in Fig. 1.6.

Now we have at least two solutions to the IVP (1.33). Thus, having extended the concept of solution to include discontinuities, extra spurious solutions are now part of this extended class. The question is how to distinguish between a physically correct solution and a spurious solution. The anticipated answer is that a physical discontinuity, in addition to the Rankine-Hugoniot condition, also satisfies the entropy condition .

**The Riemann Problem for the Inviscid Burgers Equation**

We finalise this section by giving the solution of the Riemann problem for the inviscid Burgers equation, namely

From the previous discussion the exact solution is a single wave emanating from the origin as shown in Fig. 1.7a. In view of the entropy condition this wave is either a shock wave, when , or a rarefaction wave, when . The complete solution is

Fig. 1.7 shows the solution of the Riemann problem for the inviscid Burgers equation. Fig. 1.7a depicts the structure of the general solution and consists of a single wave, Fig. 1.7b shows the case in which the solution is a shock wave and Fig. 1.7c shows the case in which it is a rarefaction wave.

Some of the studied notions for scalar conservations laws extend quite directly to systems of hyperbolic conservations laws, as we see in the next section.

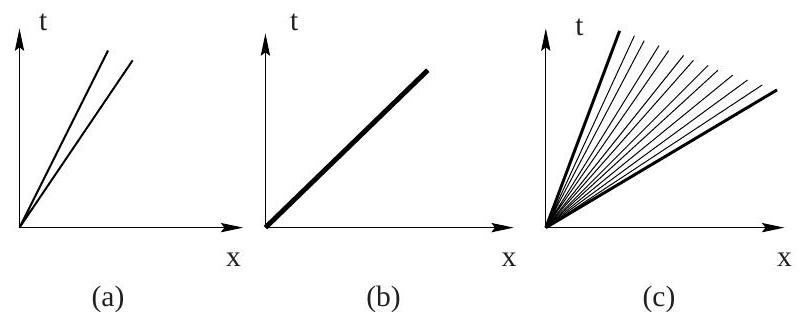


Fig. 1.7. Solution of the Riemann problem for the inviscid Burgers equation: (a) structure of general solution (single wave, shock or rarefaction), (b) solution is a shock wave and (c) solution is a rarefaction wave

**2: One Dimensional Euler Equations**

Here, we use both conservative and non-conservative formulas to examine the one-dimensional time-dependent Euler equations with an ideal Equation of State. An outline of the fundamental structure of the Riemann issue solution is provided.

* 1. **Conservative Formulation**

The conservative formulation of the Euler equations, in differential  
form, is

where and are the vectors of conserved variables and fluxes, given respectively by,

Here is density, is pressure, is particle velocity and is total energy per unit volume.

where e is the specific internal energy given by a caloric Equation of State,

Such Conservation laws can be written in Quasi Linear form as,

where,

**2.2 Non Conservative Formulation**

**2.2.1 Primitive Variable Formulation**

It is possible to formulate and solve the equations using variables other than the conserved variables in order to achieve smooth solutions.

Selecting a vector is one option for the one-dimensional scenario. So we choose

.

The proof is as follows :

We have,

where,

Now on substitution and partially differentiating, we get

and from the 2nd row, we get the second equation as,

On rearranging (2) we get,

and on substituting (2) in (3), we get

Now, from the 3rd row we obtain, and using appropriate substitution, we get

In Matrix notation, we can write it as,

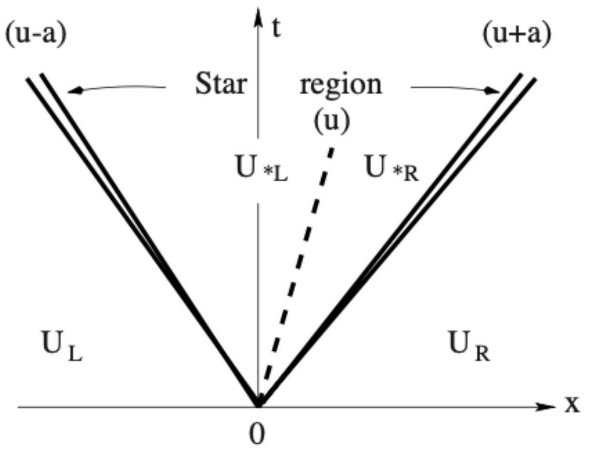
where,

One May also find the Eigen values and Eigen Vectors of this system, using , getting

On solving this the Eigen values and Eigen vectors come out to be,

**2.3 Elementary Wave Solutions to Reimann Problem**

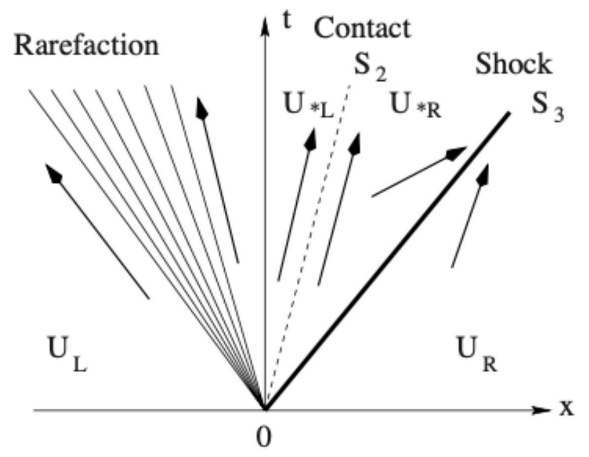
The Riemann problem for the one-dimensional, time dependent Euler equations with with data is the IVP



**Figure 1:** Structure of the solution of the Riemann problem in the plane for the time-dependent, one-dimensional Euler equations. There are three wave families associated with the eigenvalues and

The three characteristic fields that correspond to the eigenvectors , , are connected with three waves. When the character of the  
outer waves is uncertain, we follow the norm of depicting them as two rays coming from the origin for the outer waves and a dashed line for the central wave. Each wave family and its related eigenvalue are displayed. Four steady states are divided by the three waves. The wave associated with the characteristic field is a contact discontinuity and those associated with the characteristic fields will either be rarefaction waves (smooth) or shock waves (discontinuities). The only exception is the middle wave, which is always a contact discontinuity.

The Figure below shows a particular case in which the left wave is a rarefaction, the middle wave is a contact and the right wave is a shock wave.



**Figure 2:** Structure of the solution of the Riemann problem in the plane for the time-dependent, one-dimensional Euler equations, in which the left wave is a rarefaction, the middle wave is a contact discontinuity and the right wave is a shock wave

For the rarefaction wave we have, ( is the speed of shock)

For the shock wave we have,

For the contact wave we have, ( is the speed of the contact wave)

In the Upcoming sections, we look at each of the wave types separately.

**2.3.1 Contact Discontinuities**

Suppose, we have a system like,

where and are the vectors of conserved variables and fluxes, given respectively by,

or say in general with,

and the Right Eigen vectors as,

We can write the ith Generalised Reimann Invariant as the (m-1) ODEs

So in applying this on our system, we get

Manipulation gives us,

So in short, Contact wave is a discontinuous wave across which both pressure and particle velocity are constant, but density jumps discontinuously as do variables that depend on density, such as specific internal energy, temperature, sound speed, entropy, etc.

**2.3.2 Rarefaction Waves**

In the Euler equations, the and characteristic fields are related to rarefaction waves. Upon examining the eigenvectors for the primitive-variable formulation, it can be observed that , and undergo changes throughout a rarefaction wave. Using Entropy Formulation, we have

On Solving for the Eigen values and Eigen vectors, we get

For , using the Generalised Reimann Invariants,

From this we get,

Integrating the 2nd equation, we get

Now the Isentropic law states that , i.e , and we also know that , using these we get,

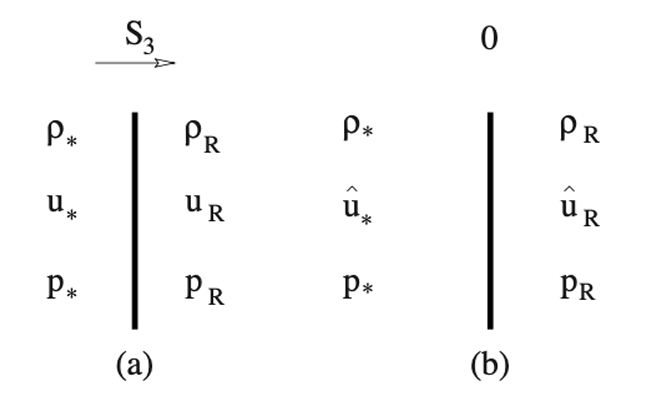
Substituting for a in equation (1), we get

So finally, we obtain,

Similarly we can get the following condition for ,

To summarise: a rarefaction wave is a smooth wave associated with the 1 and 3 fields across which , and change. The wave has a fan-type shape and is enclosed by two bounding characteristics corresponding to the Head and the Tail of the wave.

**1.3.3 Shock Waves**

Shock waves are discontinuous waves associated with the genuinely non-linear fields 1 and 3. All three quantities , and p change across a shock wave. Consider the characteristic field and assume the corresponding wave is a right-facing shock wave travelling at the constant speed S3; see Fig,

**Figure 3:** Shock wave facing right: (a) With a stationary frame of reference, the shock has a speed of S3; (b) A moving frame of reference causes the shock to have a speed of zero.

Let the state behind the shock be and the state ahead of the shock be . Densities and pressures remain unaltered while velocities have changed to the relative velocities and given by,

Applying the Rankine Hugoniot conditions, we get

Using the Definitions of Total Energy E, specific internal Energy e , specific enthalpy and caloric equations of state, followed by algrabic Modifications, we get the following Relations, (For a detailed derivation refer

Now Introducing Mach Numbers, ,

And the shock speed as a function of pressure ratio across the shock is,

The analysis is exactly the same for a 1 -shock wave (left facing) moving at velocity .

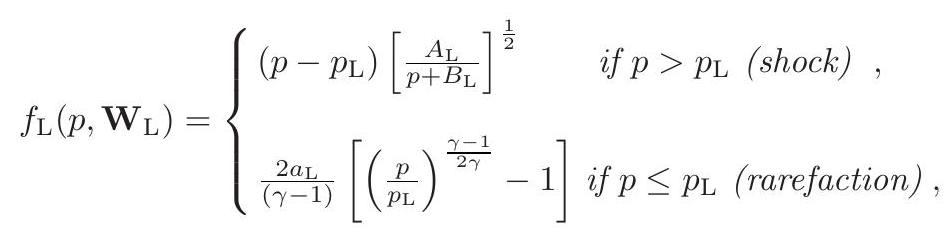
**3: The Riemann Problem for the Euler Equations**

**3.1 Equations for Pressure and Particle Velocity**

Here we establish equations and solution strategies for computing the pressure and the particle velocity in the Star Region.

**Proposition 3.1 (solution for and ).** The solution for pressure of the Riemann problem with the ideal gas Equation of State is given by the root of the algebraic equation

where the function is given by



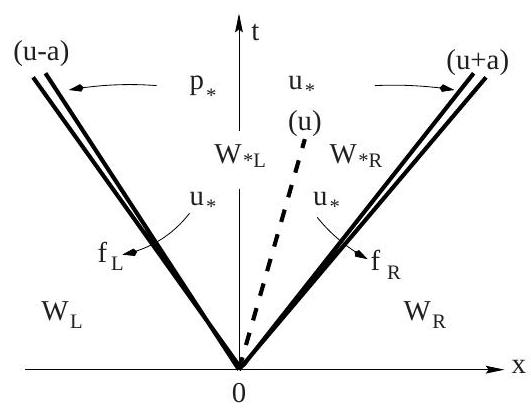
(3.2)

the function is given by

and the data-dependent constants are given by

The solution for the particle velocity in the Star Region is

Remark 3.2. Before proceeding to prove the above statements we make some useful remarks. Once (3.1) is solved for the solution for follows as in (3.9) and the remaining unknowns are found by using standard gas dynamics relations. The function governs relations across the left non-linear wave and serves to connect the unknown particle speed to the known state on the left side, see Fig. 3.3; the relations depend on the type of wave (shock or rarefaction). The arguments of are the pressure and the data state . Similarly, the function governs relations across the right wave and connects the unknown to the right data state ; its arguments are and . For convenience we shall often omit the data arguments of the functions and . The sought pressure in the Star Region is the root of the algebraic equation (4.1), . A detailed analysis of the pressure function reveals a particularly simple behaviour and that for physically relevant data there exists a unique solution to the equation .



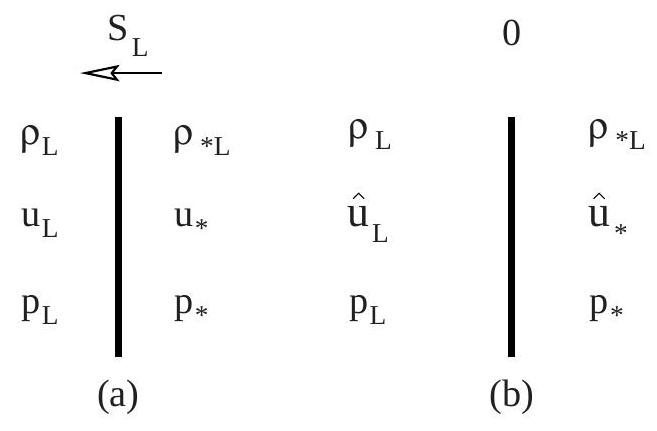
**Fig. 3.3.** Strategy for solving the Riemann problem via a pressure function. The particle velocity is connected to data on the left and right via functions and

**3.1.1 Function for a Left Shock**

We assume the left wave is a shock moving with speed as shown in Fig. 3.4a; pre-shock values are and and post-shock values are and .

As done in Sect. 3.1.3 of Chap. 3, we transform the equations to a frame of reference moving with the shock, as depicted in Fig. 3.4b. In the new frame the shock speed is zero and the relative velocities are

The Rankine-Hugoniot Conditions, see Sect. 3.1.3 of Chap. 3, give



**Fig. 3.4.** Left wave is a shock wave of speed : (a) stationary frame, shock speed is (b) frame of reference moving with speed , shock speed is zero

We introduce the mass flux , which in view of (3.7) may be written as

From equation (3.8)

Use of (3.10) and solving for gives

But from equation (3.6) and so becomes

from which we obtain

We are now close to having related to data on the left hand side. We seek to express the right hand side of (3.13) purely in terms of and , which means that we need to express as a function of and the data on the left hand side. We substitute the relations

obtained from (3.10) into equation (3.11) to produce

As seen in Sect. 3.1.3 of Chap. 3, the density is related to the pressure behind the left shock via

Substitution of into (3.14) yields

which in turn reduces (3.13) to

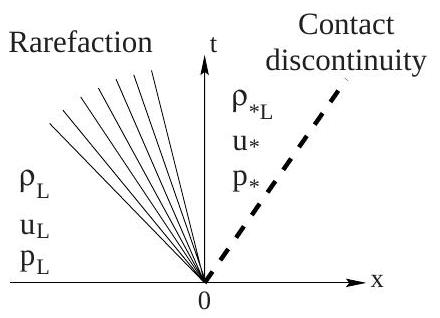
with

and

Thus, the sought expression for for the case in which the left wave is a shock wave has been obtained.

**3.1.2 Function for Left Rarefaction**

Now we derive an expression for for the case in which the left wave is a rarefaction wave, as shown in Fig. 3.5. The unknown state is now connected to the left data state using the isentropic relation and the Generalised Riemann Invariants for the left wave. As seen in Sect. 3.1.2 of



**Fig. 3.5**. Left wave is a rarefaction wave that connects the data state with the unknown state in the star region to the left of the contact discontinuity

Chap. 3, the isentropic law

where is a constant, may be used across rarefactions. is evaluated at the initial left data state by applying the isentropic law, namely

and so the constant is

from which we write

In Sect. 3.1.3 of Chap. 3 we showed that across a left rarefaction the Generalised Riemann Invariant is constant. By evaluating the constant on the left data state we write

where and denote the sound speed on the left and right states bounding the left rarefaction wave. See Fig. 3.5.

Substitution of from (3.19) into the definition of gives

and equation (3.20) leads to

with

This is the required expression for the function for the case in which the left wave is a rarefaction wave.

**3.1.3 Function for a Right Shock**

Here we find the expression for the function for the case in which the right wave is a shock wave travelling with speed . The situation is entirely analogous to the case of a left shock wave. Pre-shock values are and and post-shock values are and . In the transformed frame of reference moving with the shock, the shock speed is zero and the relative velocities are

Application of the Rankine-Hugoniot conditions gives

Now the mass flux is defined as

By performing algebraic manipulations similar to those for a left shock we derive the following expression for the mass flux

Hence the particle velocity in the Star Region satisfies

with

This is the sought expression for for the case in which the right wave is a shock wave.

**3.1.4 Function for a Right Rarefaction**

The derivation of the function for the case in which the right wave is a rarefaction wave is carried out in an entirely analogous manner to the case of a left rarefaction. The isentropic law gives

and the Generalised Riemann Invariant for a right rarefaction gives

Using (3.28) into the definition of sound speed gives

which if substituted into (3.27) leads to

with

The functions and have now been determined for all four possible wave patterns of Fig. 3.2. Now by eliminating from equations (3.17) or (3.22) and (3.27) or (3.29) we obtain a single equation

which is the required equation (3.5) for the pressure. This proves the first part of the proposition. Assuming this single non-linear algebraic equation is solved (numerically) for then the solution for the particle velocity can be found from equation (3.17) if the left wave is a shock or from equation (3.22) if the left wave is a rarefaction ( ) or from equation (3.27) if the right wave is a shock or from equation (3.29) if the right wave is a rarefaction wave . It can also be found from a mean value as

which is equation (3.9), and the proposition has thus been proved.

**3.2 Numerical Tests**

Five Riemann problems are selected to test the performance of the Riemann solver and the influence of the initial guess for pressure. The tests are also used to illustrate some typical wave patterns resulting from the solution of the Riemann problem. Table 3.1 shows the data for all five tests in terms of primitive variables. In all cases the ratio of specific heats is .

Test 1 is the so called Sod test problem; this is a very mild test and its solution consists of a left rarefaction, a contact and a right shock. Fig. 3.6 shows solution profiles for density, velocity, pressure and specific internal energy across the complete wave structure, at time units. Test 2 , called the 123 problem, has solution consisting of two strong rarefactions and a trivial stationary contact discontinuity; the pressure is very small (close to vacuum) and this can lead to difficulties in the iteration scheme to find numerically. Fig. 3.7 shows solution profiles. Test 2 is also useful in assessing the performance of numerical methods for low density flows. Test 3 is a very severe test problem, the solution of which contains a left rarefaction, a contact and a right shock; this test is actually the left half of the blast wave problem of Woodward and Colella, Fig. 3.8 shows solution profiles. Test 4 is the right half of the Woodward and Colella problem; its solution contains a left shock, a contact discontinuity and a right rarefaction, as shown in Fig. 3.9. Test 5 is made up of the right and left shocks emerging from the solution to tests 3 and 4 respectively; its solution represents the collision of these two strong shocks and consists of a left facing shock (travelling very slowly to the right), a right travelling contact discontinuity and a right travelling shock wave. Fig. 3.10 shows solution profiles for Test 5.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Test |  |  |  |  |  |  |
| 1 | 1.0 | 0.0 | 1.0 | 0.125 | 0.0 | 0.1 |
| 2 | 1.0 | -2.0 | 0.4 | 1.0 | 2.0 | 0.4 |
| 3 | 1.0 | 0.0 | 1000.0 | 1.0 | 0.0 | 0.01 |
| 4 | 1.0 | 0.0 | 0.01 | 1.0 | 0.0 | 100.0 |
| 5 | 5.99924 | 19.5975 | 460.894 | 5.99242 | -6.19633 | 46.0950 |

**Table 3.1.** Data for five Riemann problem tests

**Table 3.2** shows the computed values for pressure in the Star Region by solving the pressure equation (equation 3.5) by a Newton-Raphson method. This task is carried out by the subroutine STARPU, which is contained in the FORTRAN 77 program given in Sect. 3.9 of this chapter.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Test |  |  |  |  |  |
| 1 | 0.30313 |  |  |  |  |
| 2 | 0.00189 | exact | TOL | TOL |  |
| 3 | 460.894 |  |  |  |  |
| 4 | 46.0950 |  |  |  |  |
| 5 | 1691.64 |  |  |  |  |

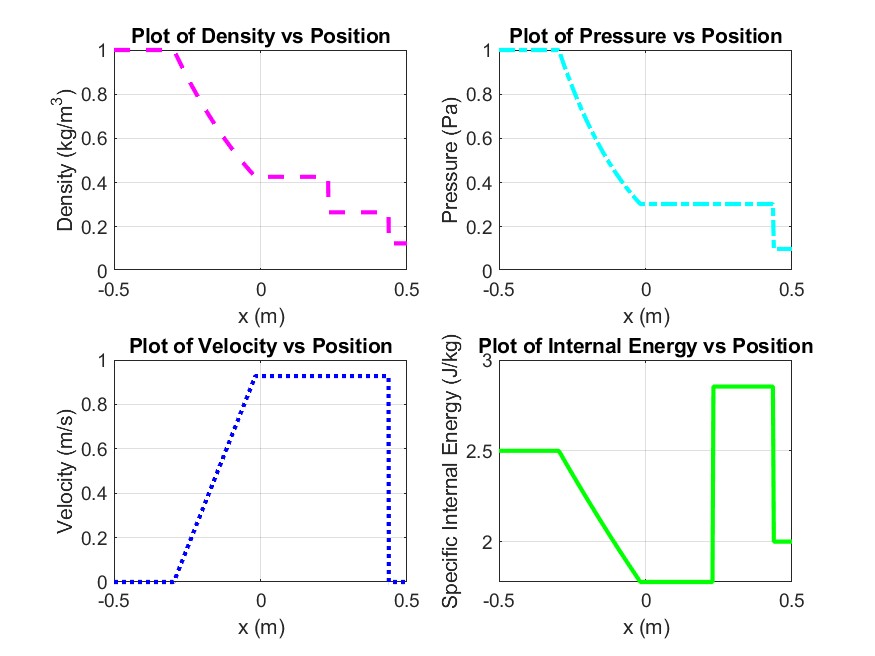
**Table 3.2** Guess values for iteration scheme. Next to each guess is the required number of iterations for convergence (in parentheses).

and . In a typical application of the exact Riemann solver to a numerical method, the overwhelming majority of Riemann problems will consist of nearby states which can be accurately approximated by the simple value .

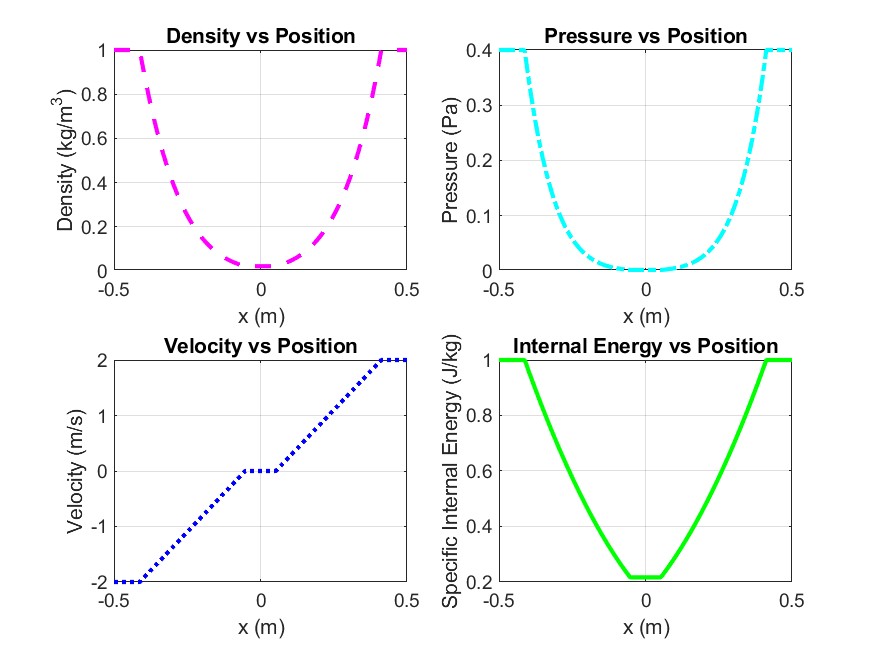
Having found , the solution for the particle velocity follows from (3.9) and the density values follow from appropriate wave relations, as detailed in the next section. Table 3.3 shows exact solutions for pressure , speed , densities and for tests 1 to 5 . These quantities may prove of some use for initial testing of programs.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Test |  |  |  |  |
| 1 | 0.30313 | 0.92745 | 0.42632 | 0.26557 |
| 2 | 0.00189 | 0.00000 | 0.02185 | 0.02185 |
| 3 | 460.894 | 19.5975 | 0.57506 | 5.99924 |
| 4 | 46.0950 | -6.19633 | 5.99242 | 0.57511 |
| 5 | 1691.64 | 8.68975 | 14.2823 | 31.0426 |

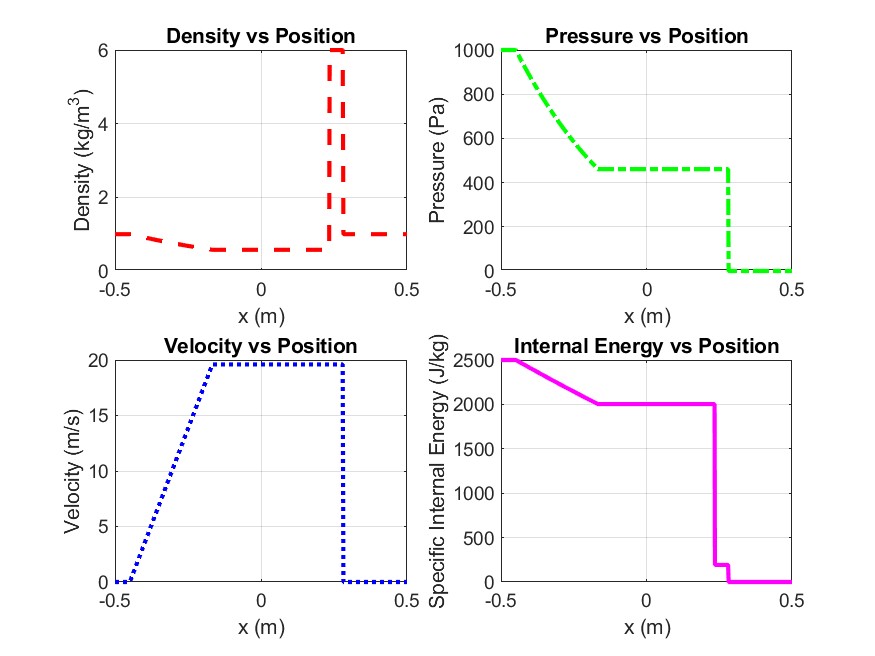
**Table 3.3.** Exact solution for pressure, speed and densities for tests 1 to 5.



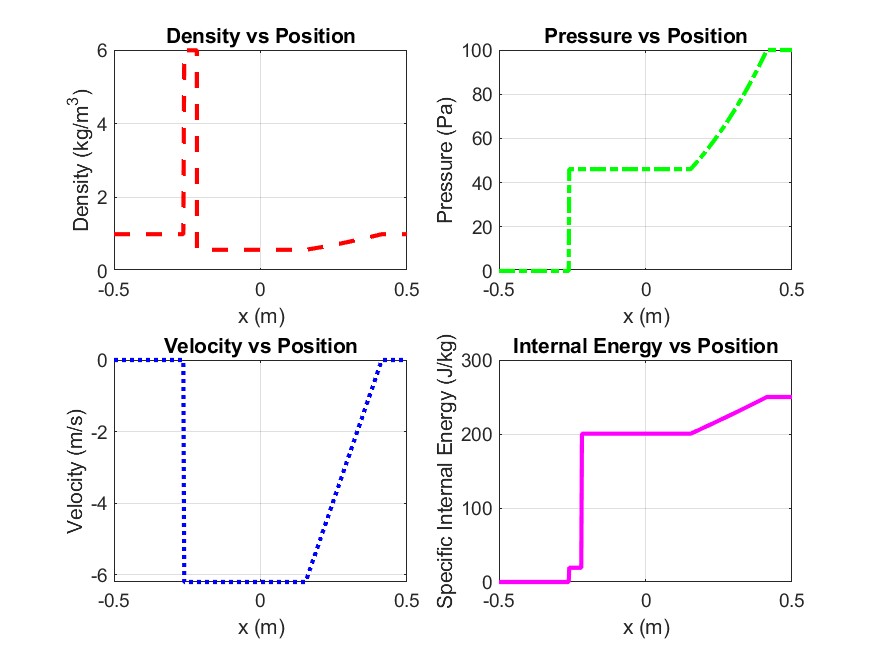
**Fig 3.6** Plot produced using MATLAB for Test-1



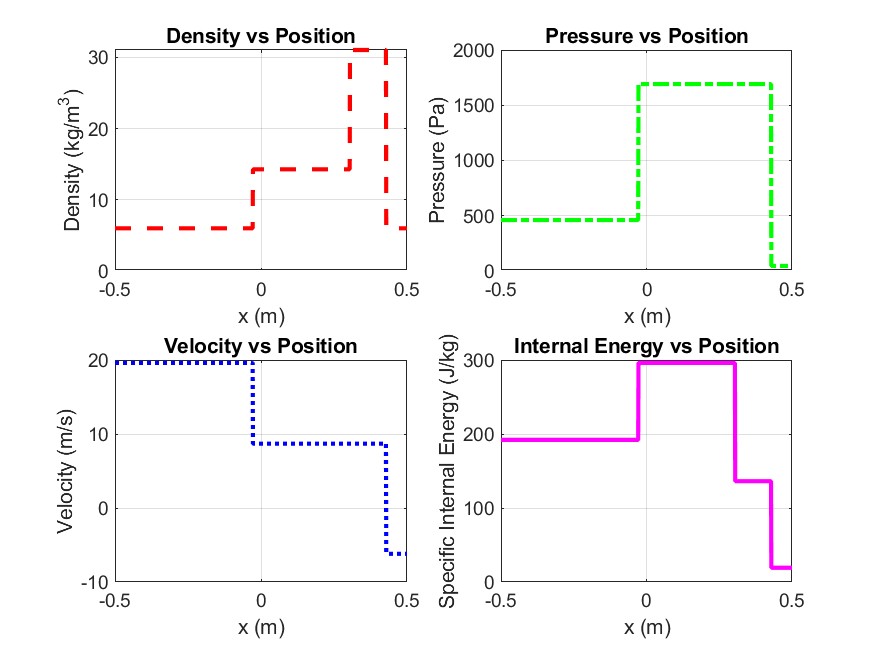
**Fig 3.7** Plot produced using MATLAB for Test-2



**Fig 3.8** Plot produced using MATLAB for Test-3



**Fig 3.9** Plot produced using MATLAB for Test-4



**Fig 3.10** Plot produced using MATLAB for Test-5

**4: Numerical Solution Using Iterative Schemes**

**4.1 Some Well-Known Schemes**

Another first-order scheme is that of Lax and Friedrichs. The scheme is sometimes also called the Lax Method, or the scheme of Keller and Lax. This does not require the differencing to be performed according to upwind directions and can be seen as a way of stabilizing the unstable scheme obtained from forward in time and central in space approximations to the partial derivatives. The **Lax-Friedrichs** scheme results if in the time derivative is replaced by

a mean value of the two neighbors at time level . Then the modified scheme becomes

or

A von Neumann stability analysis reveals that scheme (4.1) is stable under the stability condition and a truncation error analysis says that the scheme is first-order accurate. The modified equation is like with numerical viscosity coefficient given by

By comparing with we see that the **Lax-Friedrichs** scheme is considerably more diffusive than the CIR scheme; in fact for we have

When written in the form the coefficients of the **Lax-Friedrichs** scheme are found to be

Under the stability condition all coefficients in the **Lax-Friedrich** scheme (4.1) are positive or zero. Therefore the scheme is monotone.

A scheme of historic as well as practical importance is that of Lax and Wendroff [302]. For a comprehensive treatment of the family of **Lax-Wendroff**. The basic **Lax-Wendroff** scheme is second-order accurate in both space and time. There are several ways of deriving the scheme for the model equation in (4.7). A rather unconventional derivation is this: for the time derivative insist on the first-order forward approximation (4.9); for the space derivative take an average of the upwind (stable if ) and downwind (unstable if ) approximations (4.14) and (4.15) respectively, that is

If the coefficients are chosen as

the resulting scheme is the **Lax-Wendroff** method

This scheme is second-order in space and time although all finite difference approximations used to generate it are first-order accurate. Moreover, one of the terms in the spatial derivative originates from an unconditionally unstable scheme and yet the **Lax-Wendroff** scheme is stable with stability condition. This scheme is a good example to show that the order of accuracy of the scheme cannot in general be inferred from the order of accuracy of the finite difference approximations to the partial derivatives involved.

When written in the form the **Lax-Wendroff** scheme (4.5) has coefficients

Therefore this scheme is not monotone. Not all coefficients in (4.5) are positive or zero. The fact that a scheme is not monotone is associated with the phenomenon of spurious oscillations in the numerical solution in the vicinity of sharp gradients, such as at discontinuities

Another second-order accurate scheme for (4.7) is the upwind method of **Warming and Beam**. For positive speed it reads

Note that the scheme is fully one-sided in the sense that all the mesh points involved, other than the centre of the stencil, are on the left hand side of the centre of the stencil. There is an equivalent scheme for negative speed . Clearly the Warming-Beam scheme is not monotone. The stability restriction for this scheme is

The enlarged stability range means that one may advance in time with a larger time step , which has a bearing on the efficiency of the scheme.

Second-order schemes such as the **Lax-Wendroff** and **Warming-Beam** and have modified equation of the form

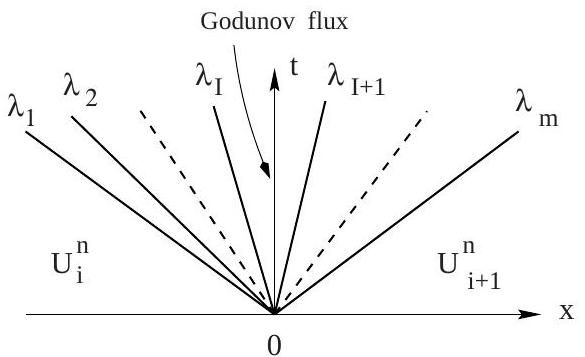
which is a dispersive equation.

**4.2 Godunov's Method**

Consider the constant coefficient, linear hyperbolic system written in conservation-law form

The Godunov first-order upwind method utilises the conservative formula (4.76) and requires the solution of the local Riemann problem for (4.10) to compute the intercell numerical flux

Here is the value of the solution at along the intercell boundary. As seen in Sect. 2.3.3 of Chap. 2, the solution



**Fig. 4.1**. Evaluation of the Godunov intercell flux for linear hyperbolic systems with constant coefficients

can be easily found by first expanding the initial data in terms of the right eigenvectors as

The general solution at any point is given by

where is the largest integer with such that . The Godunov flux (4.11) requires the solution at in (4.13). See Fig. 4.1. For is such that and , then is obtained by manipulating (4.13), namely

or

Recall that the jump across wave with eigenvalue and eigenvector is given by . Note that the solution of the Riemann problem, at , as given by (4.14), can be interpreted as being the left data state plus all wave jumps across waves of negative or zero speed. Similarly, the form (4.15) gives the solution as the right data state minus the wave jumps across all waves of positive or zero speeds. By combining (4.14) and (4.15) we obtain

The Godunov intercell numerical flux (4.11) can now be obtained by evaluating at any of the expressions (4.14)-(4.16) for the solution of the Riemann problem. Use of (4.14) gives

and since ,

Similarly, (4.15) gives

or combining (4.18) and (4.19) we obtain

Next we show that the Godunov flux can also be expressed in two more alternative forms.

Proposition 4.20. The Godunov flux (4.11) to solve (4.10) via (4.76) can be written as

Proof. Starting from (4.20) and using the properties one writes

Hence

and the proposition is proved.

**Proposition 4.1.** The Godunov flux (4.11) for (4.10) can be written in flux-split form as

Proof. The result follows directly from manipulating (4.21) and using appropriate definitions. Alternatively we have

and the proposition is proved.

**Remark 4.1**. The intercell flux has been split as

where the positive and negative flux components are

Note that, trivially, the respective Jacobian matrices have eigenvalues that are all positive (or zero) and all negative (or zero).

*Exercise 4.1.* Consider the linearised equations of Gas Dynamics

with

* Find the matrices .
* Write the scheme (4.93) in full, that is for the two components of the vector of unknowns.
* Compute the Godunov intercell flux directly by using the explicit solution of the Riemann problem in the Star Region

Write a computer program to solve the linearised equations of Gas Dynamics using the method of Godunov.

**4.3 Sample Numerical Results**

To complete this chapter, we present some numerical results obtained by some of the most well known schemes as applied to two model PDEs.

**4.3.1 Linear Advection**

We apply four schemes to solve

with two types of initial conditions.

**Test 1 for linear advection (smooth data)**

Here the initial condition is the smooth profile

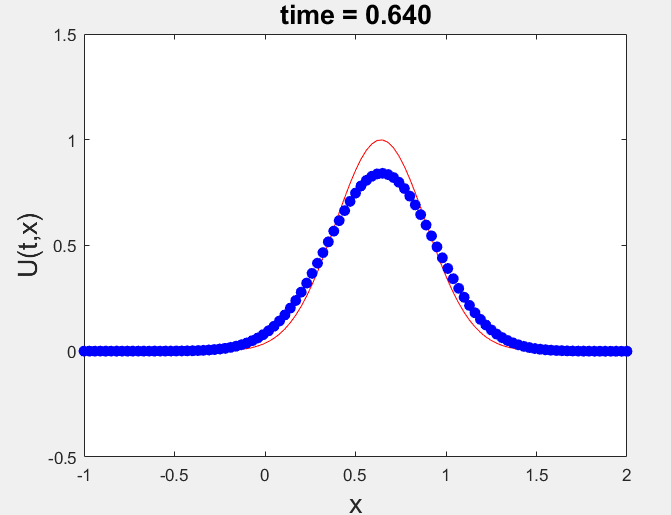
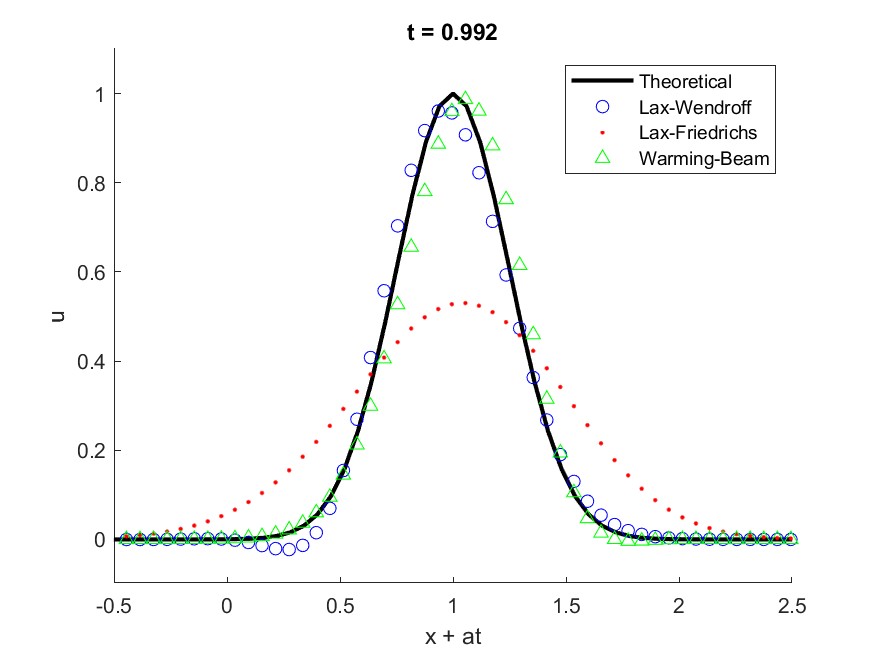
In the computations we take and a CFL coefficient ; the initial profile is evaluated in the interval . Computed results are shown in Figs. 4.9 to 4.11; these correspond respectively to the output times unit (125 time steps), units (1250 time steps), units (12499 time steps). In each figure we compare the exact solution (shown by full lines) with the numerical solution (symbols) for the Godunov method, the **Lax-Friedrichs** method, the **Lax-Wendroff** method and the Warming-Beam method.

The results of Fig. 4.9 are in many ways representative of the quality of each scheme. Collectively these results are also representative of most of the current successes and limitations of numerical methods for PDEs governing wave propagation. The first-order method of Godunov (CIR scheme) has modified equation of the form (4.24), where is a numerical viscosity coefficient. This is responsible for the clipping of the peak values. As seen earlier , which explains the fact that the **Lax-Friedrichs** scheme gives even more diffused results. For the computational parameters used and .

The results from the **Lax-Wendroff** method and the Warming-Beam method, both second-order accurate, are much more accurate than those of the first-order schemes. There are however, slight signs of error in the position of the wave. For the **Lax-Wendroff** scheme the computed wave lags behind the true wave (lagging phase error), while for the Warming-Beam method the computed wave is ahead of the true wave (leading phase error). The phase errors of second-order accurate schemes are explained by the dispersive term of the modified equation (4.9).

The limitations of the schemes are more clearly exposed if the solution is evolved for longer times. Fig. 4.10 shows results at the output time 10.0 units (1250 time steps). Compare with Fig. 4.9. The numerical diffusion inherent in first-order methods has ruined the solution of the Godunov and **Lax-Friedrichs** schemes. Computed peak values are only of the order of 30 to of the true peak values. The second-order methods are still giving more satisfactory results than their first-order counterparts, but now the numerical dispersion errors are clearly visible. Numerical diffusion is beginning to show its effects too.

Fig. 4.11 shows results at the output time units (12499 time steps). Compare with Figs. 4.9. and 4.10. These results are truly disappointing and clearly expose the limitations of numerical methods for computing solutions to problems involving long time evolution of wave phenomena. In acoustics one may require the computation of (i) very weak signals (ii) over long distances. The combination of these two requirements rules out automatically a wide range of otherwise acceptable numerical methods for PDEs. See Tam and Webb [480]. The numerical diffusion of the first-order schemes has virtually flattened the wave, while the numerical dispersion of the second order methods has resulted in unacceptable position errors, in addition to clipping by numerical ffusion.



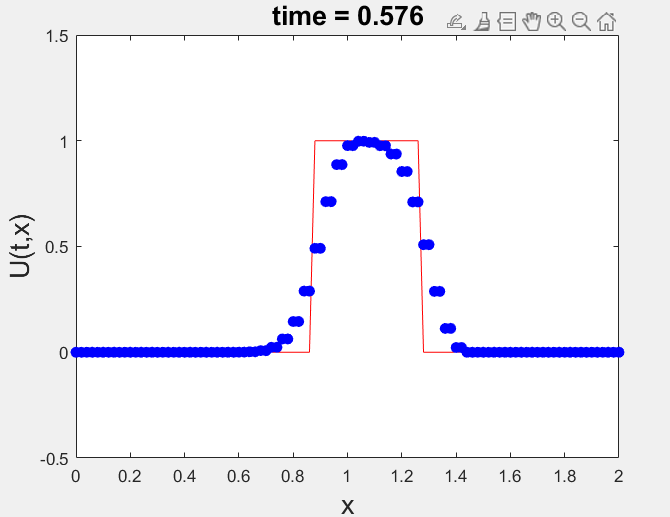
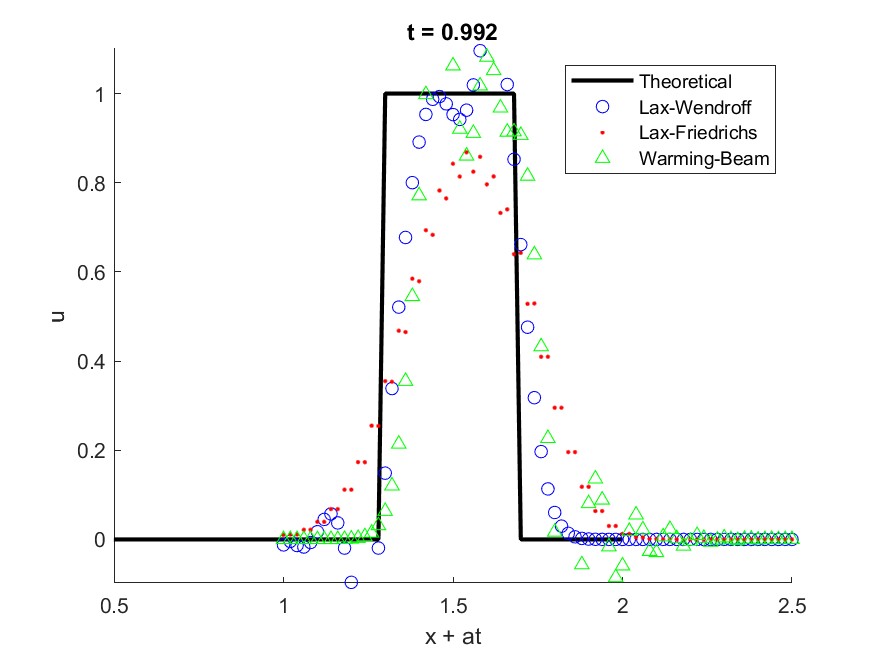
**Fig 4.2** Plot produced using MATLAB for linear advection equation for smooth data at the given time

**Test 2 for linear advection (discontinuous data)**

Now the initial data for (4.25) consists of a square wave, namely

The computed results for the three output times are shown in Figs. 4.12 to 4.14. As for Test 1 the effects of numerical diffusion in the first-order methods and the effects of dispersion in the second-order methods lead to visible errors in the numerical solution (symbols), as compared with the exact solution (full line). First-order methods smear discontinuities over many computing cells; as expected this error is more apparent in the **Lax-Friedrichs** scheme. Note also the pairing of neighbouring values in the **Lax-Friedrichs** scheme. Second-order methods reduce the smearing of discontinuities, but at the cost of overshoots and undershoots in the vicinity of the discontinuities. These spurious oscillations are highly undesirable features of second and higherorder methods. We shall return to this theme in Chaps. 13 and 14, where improved methods for dealing with discontinuities will be presented.

Fig. 4.13 shows results for Test 2 at time units (1250 time steps). The errors observed in Fig. 4.12 are now exaggerated. Fig. 4.14 shows results at time units (12499 time steps). Once again first-order methods have lost the solution while second-order methods exhibit unacceptable position errors, in addition to spurious oscillations



**Fig 4.3** Plot produced using MATLAB for linear advection equation for discontinuous data at the given t time



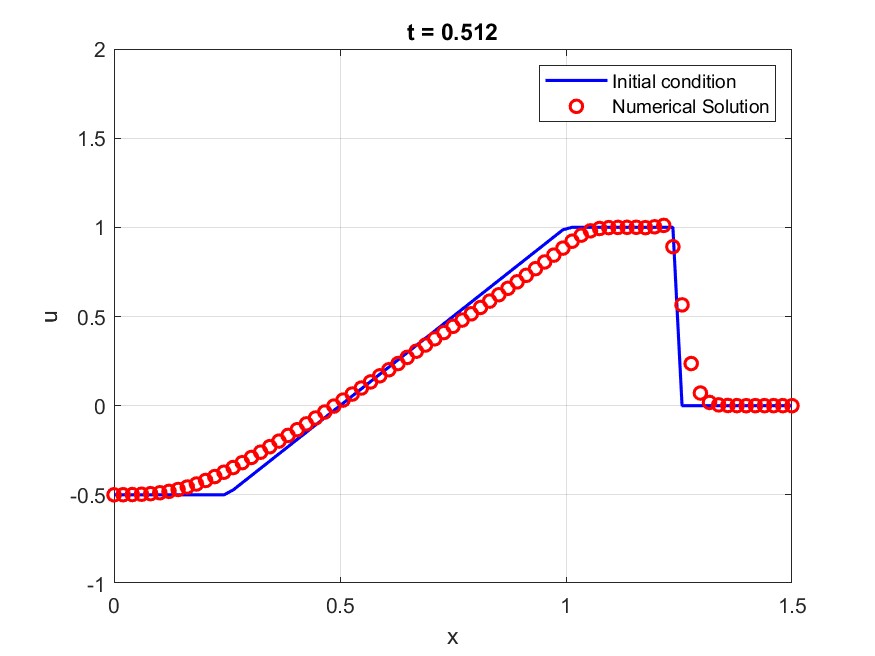
**4.3.2 The Inviscid Burgers Equation**

Our Test 3 consists of the inviscid Burgers equation

in the domain with initial conditions

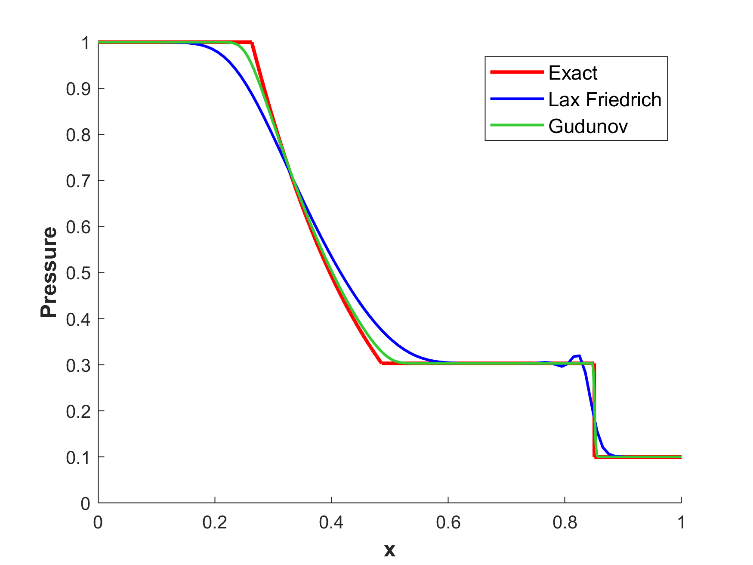
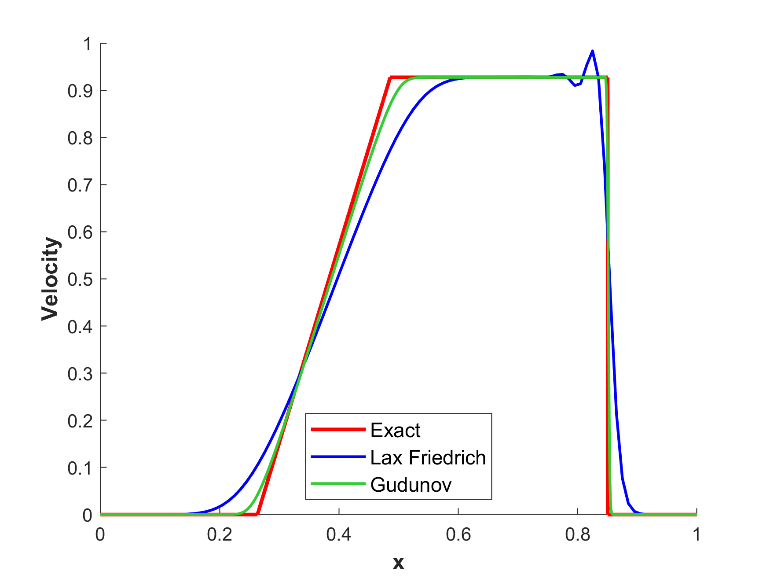
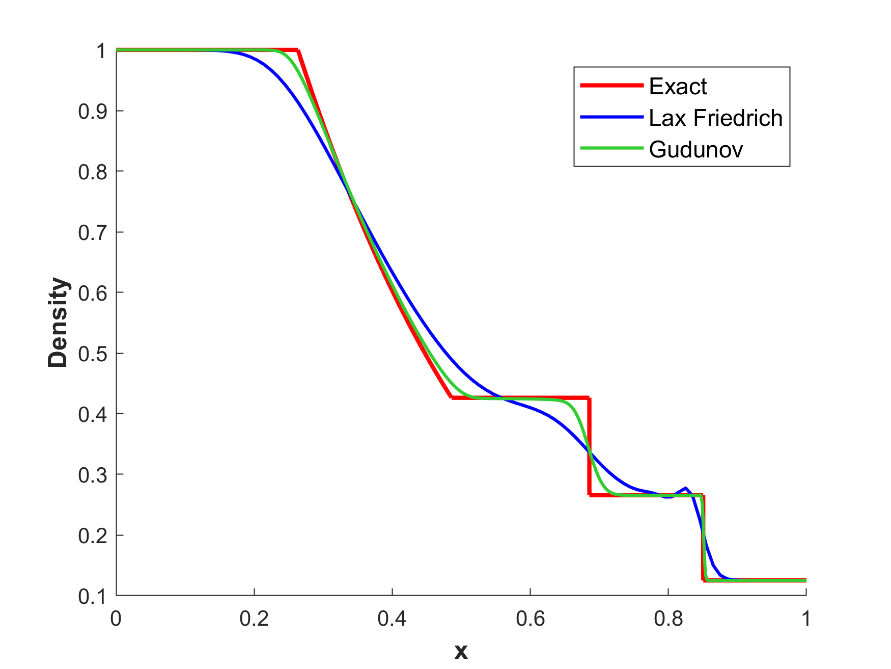
We solve this problem numerically on a domain of length discretised by equally spaced cells of width ; the CFL coefficient used is 0.8 . Fig. 4.15 shows computed results (symbol) along with the exact (line) solution, for the Godunov and **Lax-Friedrichs** schemes at time units (32 time steps). Two new features are now present in solving non-linear PDEs. First the discontinuity on the right is a shock wave. This satisfies the entropy condition, see Sect. 2.4.2 of Chap. 2, and characteristics on either side of the discontinuity converge into the discontinuity. This compression mechanism helps the more accurate resolution of shock waves. Compare with Fig. 4.12. The Godunov method resolves the shock much more sharply ( 3 cells) than the **Lax-Friedrichs** scheme (10 cells). The second new feature to note in this

non-linear example is the entropy glitch at . The entropy glitch affects the Godunov method and not the **Lax-Friedrichs** method. A question of crucial importance is the construction of entropy satisfying schemes .



**Fig 3.8** Plot produced using MATLAB for Inviscid Burgers Equation

**4.4 Numerical Scheme Solutions for Riemann Problem**

Performing the Godunov’s scheme and Lax-Friedrichs scheme on Test-1 discussed earlier in Chapter 3 which is the so called Sod test problem; this is a very mild test and its solution consists of a left rarefaction, a contact and a right shock. Figure below shows solution profiles for density, velocity, pressure and specific internal energy across the complete wave structure, at time units.

**5: Conclusion**

In conclusion, this project has been an enriching voyage into the realm of computational fluid dynamics, delving deeply into the intricacies of Riemann problems, shock waves, and rarefactions. Through meticulous study and hands-on implementation, I have garnered a profound understanding of various numerical methodologies, including Godunov's, Lax-Friedrichs, and Lax-Wendroff methods. By employing these techniques in MATLAB, I have gained practical insights into their behaviors, strengths, and limitations when applied to solving the linear advection equation.

Recognizing Riemann problems as fundamental to grasping wave behaviors, particularly shocks and rarefactions, underscores the criticality of employing accurate numerical approaches to capture these phenomena effectively. Moreover, juxtaposing exact solutions against numerical approximations has underscored the inherent trade-offs between computational efficiency and precision inherent in these methods.

In today's era, where computational simulations drive innovations across diverse sectors such as climate modeling and aerospace engineering, the significance of robust numerical methodologies in predicting complex wave behaviors cannot be overstated. The skills and knowledge acquired from this project resonate deeply with the current demand for precise and efficient simulations in our rapidly evolving technological landscape.

I express heartfelt gratitude to Professor P Minhajul of the Department of Mathematics, BITS Pilani KK Birla Goa Campus, for his invaluable guidance, unwavering support, and the opportunity to work under his mentorship. His expertise and encouragement have been instrumental in shaping this project and fostering my growth in the field of Computational Fluid Dynamics.

**6: References**

[1] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer Berlin, Heidelberg, 3 edition, 2009.